From Dirac's Delta to VIX Indices

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Inspiration from the Brightest Physicist Ever

" ... it turned out that the equations that really work in describing nature with the most generality and the greatest simplicity are very elegant and subtle. It's the kind of beauty that might be hard to explain [but] is just as real to anyone who's experienced it as the beauty of music."

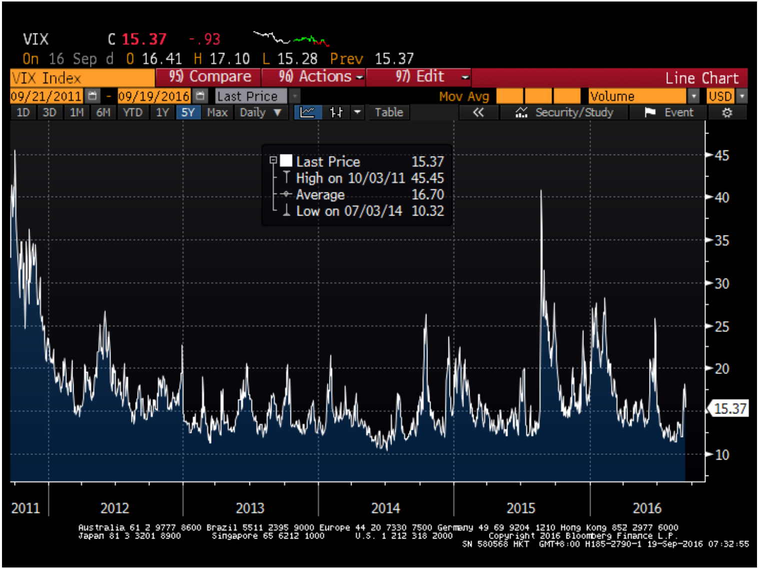
Source: Viewpoints on String Theory

Source: Edward Witten's 2014 Kyoto Prize Commemorative Lecture in Basic Sciences

Open Mind

Can some math from a theoretical physicist (Dirac) be relevant to $\mathbb{Q}$uantitative Finance?

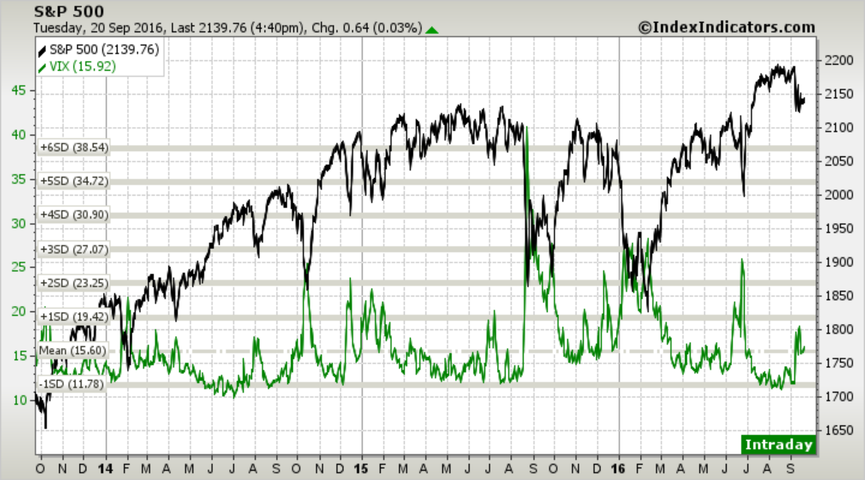
VIX: Fear Gauge?



What is VIX?

- Created in 1993 VIX is the ticker symbol for the CBOE Volatility Index for S&P 500 Index.
- VIX quantifies option traders' expectation of *future* volatility for the next 30 calendar days.
- The old version of VIX relied on the Black-Scholes model to back out an implied volatility for each of the 8 options that are near-the-money. Old VIX is the average of these implied volatilities.
- Current new version is model free, and uses as many out-of-the-money S&P 500 index options as possible.
- The formula for model-free VIX is beautiful.
- Why called the "fear gauge"?
  - Contributions of OTM put options are larger than OTM call options,
  - S&P 500 Index tends to be a lot lower when VIX is higher.

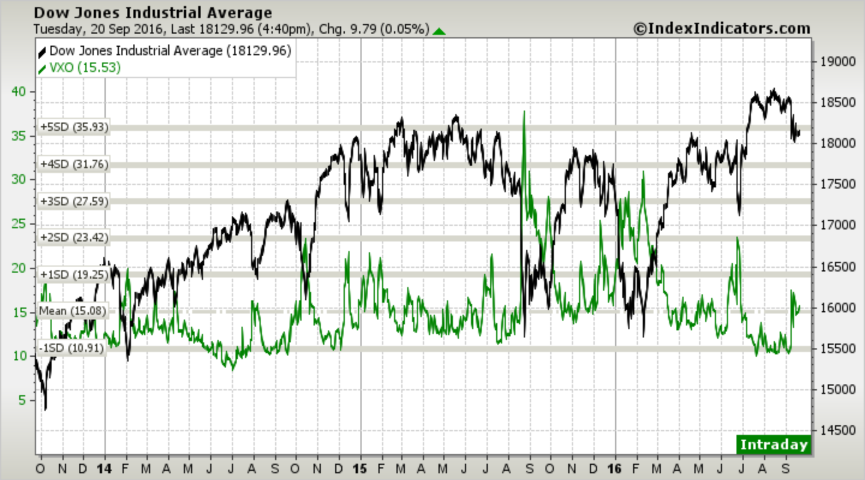
Relation with the Underlying S&P 500 Index



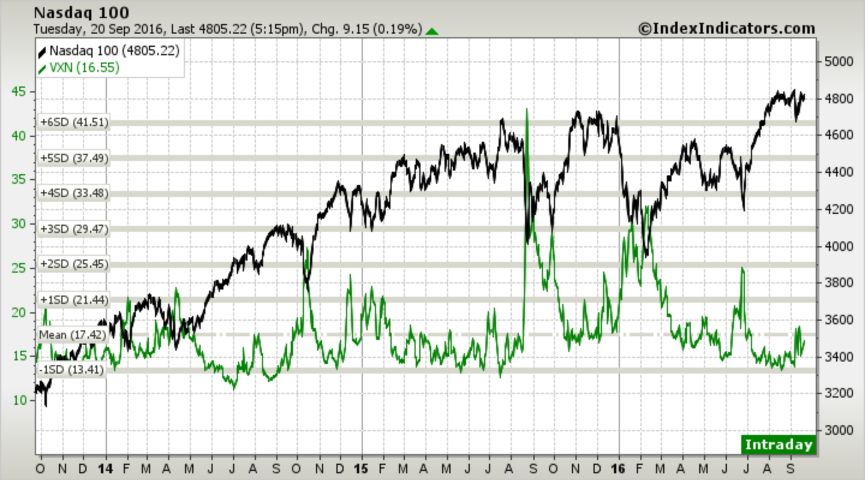
VIX: A Success Story

- Speculation on the future level of volatility in a pure manner
  - short the VIX futures when VIX is "unusually" high
  - long the VIX futures when VIX is "unusually" low
- Hedge against long equity exposure
- Hedge against a high correlation market condition, which typically makes stock selection more difficult
- Tracking of aggregate credit spread
- Tracking of carry trade benchmark

VXO and Dow Jones Industrial Average Index



VXN and Nasdaq 100 Index



Applications of VIX

- Volatility becomes a tradable "asset class".
  - CBOE offers futures and options on VIX—revenue generation for the exchange.
  - Speculation: Express a view on future volatility through trading.
  - Hedging: Reduction of NAV fluctuation.
- VIX^2 as the fair value for a 30-day variance swap. The payoff function (same as P&L in this case) of this forward contract for the buyer is, at maturity $T = 30$,

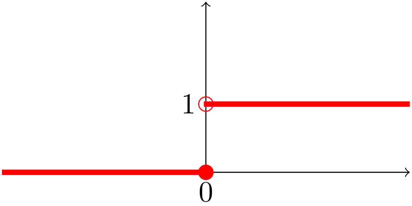
$$\text{P\&L of Buyer}_T = A \times \left(\text{Realized Variance - VIX}^2\right),$$

where the realized variance is the variance of future daily returns from day 0 up to T.

What is a Unit Step Function?

Definition

$$1_{x>0} := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$



Mathematical identity

$$1_{x>0} + 1_{x\leq 0} = 1.$$

Integration of Step Function

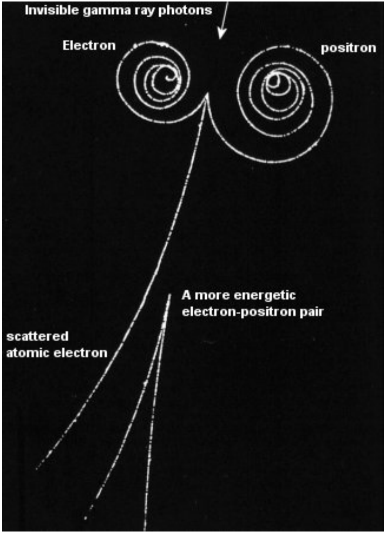
- There is nothing sacrosanct about $(0, 0)$. You can shift the origin to a non-zero number.
- Recall that $x^+ := \max(x, 0)$.
- Let λ and a be a positive real number. Then

$$\int_{-\infty}^{\lambda} 1_{x>a}\, dx = (x-a)^+ \Big|_0^{\lambda} = (\lambda - a)^+.$$

- Likewise

$$\int_{\lambda}^{\infty} 1_{x\le a}\, dx = (a-x)^+ \Big|_{\lambda}^{\infty} = (a - \lambda)^+.$$

Anti-Electron: Positron



Picture source: Bubble Chamber

Dirac's equation

$$i\hbar\gamma^{\mu}\partial_{\mu}\psi = mc\,\psi \qquad (1)$$

Which One is Paul Adrien Maurice Dirac?

Picture source: Paul Dirac and the religion of mathematical beauty

Mathematics and Beauty

We must admit that religion is a jumble of false assertions, with no basis in reality. The very idea of God is a product of the human imagination.

— Dirac (1927), atheist

Source: Physics and Beyond : Encounters and Conversations (1971) by Werner Heisenberg, pp. 85-86

God used beautiful mathematics in creating the world.

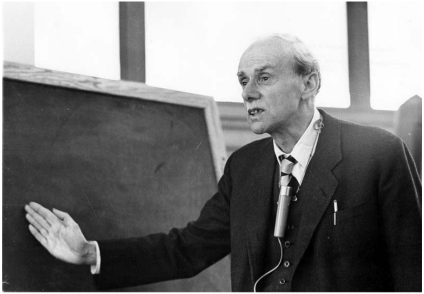
— Dirac (1963), ex-atheist

Source: The Cosmic Code: Quantum Physics As The Language Of Nature (2012) by Heinz Pagels, pp. 295

Conversion of an Ex-Atheist by Mathematical Beauty

"God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe."

The Evolution of the Physicist's Picture of Nature
Scientific American, May 1963



Paul A. M. Dirac (1962)
from AIP Emilio Segre Visual Archives

Source: http://ysfine.com/dirac/dirac44.jpg

Dirac's Delta "Function"

- Definition

$$\left.\begin{aligned} \int_{-\infty}^{\infty} \delta(x)dx &= 1 \\ \delta(x) &= 0 \qquad \text{for } x \neq 0. \end{aligned}\right\} \tag{2}$$

- The most important property of $\delta(x)$ is exemplified by the following equation,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0), \tag{3}$$

where $f(x)$ is any continuous function of x.

Dirac's Delta "Function" (Cont'd)

- By making a shift of origin to a for Dirac's δ function, we can deduce the formula

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a). \tag{4}$$

- The process of multiplying a function of x by $\delta(x-a)$ and integrating over all x is equivalent to the process of substituting a for x.

- The range of integration need not be from $-\infty$ to ∞. Any domain, say the interval $(-g_2, g_1)$ containing the critical point at which $\delta(x)$ does not vanish, will do.

Dirac's Alternative Definition of $\delta(x)$

- Consider the differential coefficient $\epsilon'(x)$ of the step function $\epsilon(x)$ given by

$$1_{x>0} := \left\{ \begin{array}{ll} \epsilon(x) = 1 & \text{if } x > 0 \\ \epsilon(x) = 0 & \text{if } x \leq 0. \end{array} \right\} \qquad (5)$$

- Substitute $\epsilon'(x)$ for $\delta(x)$ in the left side of (3). For positive g_1 and g_2, integration by parts leads to

$$\begin{aligned} \int_{-g_2}^{g_1} f(x)\epsilon'(x)dx &= f(x)\epsilon(x)\Big|_{-g_2}^{g_1} - \int_{-g_2}^{g_1} f'(x)\epsilon(x)dx \\ &= f(g_1) - \int_0^{g_1} f'(x)dx \\ &= f(0) \end{aligned}$$

Dirac's Delta Function in $\mathbb{Q}$F

- For any payoff function $f(S)$, the origin-shifting property of the Dirac function allows us to write, for any non-negative λ,

$$
\begin{aligned}
f(S) &= \int_0^\infty f(K)\delta(K-S)dK \\
&= \int_0^\lambda f(K)\delta(K-S)dK + \int_\lambda^\infty f(K)\delta(K-S)dK
\end{aligned}
$$

- Integrating each integral by parts results in

$$
\begin{aligned}
f(S) = f(K)1_{S<K}\Big|_0^\lambda &- \int_0^\lambda f'(K)1_{S<K}dK \\
&+ f(K)1_{S\geq K}\Big|_\lambda^\infty - \int_\lambda^\infty f'(K)1_{S\geq K}dK
\end{aligned}
$$

Replication by Bonds and Options

→ Integrating each integral by parts once more!

$$
\begin{aligned}
f(S) &= f(\lambda)1_{S<\lambda} - f'(K)(K-S)^+\Big|_0^\lambda + \int_0^\lambda f''(K)(K-S)^+ dK \\
&\quad + f(\lambda)1_{S\geq\lambda} - f'(K)(S-K)^+\Big|_\lambda^\infty + \int_\lambda^\infty f''(K)(S-K)^+ dK \\
&= f(\lambda) + f'(\lambda)\left[(S-\lambda)^+ - (\lambda-S)^+\right] \\
&\quad + \int_0^\lambda f''(K)(K-S)^+ dK + \int_\lambda^\infty f''(K)(S-K)^+ dK. \\
&= f(\lambda) + f'(\lambda)(S-\lambda) \\
&\quad + \int_0^\lambda f''(K)(K-S)^+ dK + \int_\lambda^\infty f''(K)(S-K)^+ dK. \qquad (6)
\end{aligned}
$$

Static Replication

- The payoff $f(S)$ contingent on the outcome S at maturity T can be replicated by
  - $f(\lambda)$: number of risk-free discount bonds, each paying \$1 at T
  - $f'(\lambda)$: number of forward contracts with delivery price λ
  - $(K - S)^+$: European put option's payoff at T of strike K
  - $(S - K)^+$: European call option's payoff at T of strike K
  - $f''(\lambda)dK$ is the number of put options of all strikes $K < \lambda$, and call options of all strikes $K > \lambda$

- The payoff replication is static, and model-free of Type 1.

Log Contract

- Let $f(x) = \log(x)$. Then $f'(x) = \frac{1}{x}$, and $f''(x) = -\frac{1}{x^2}$. We set $S = S_T$, and $\lambda = F_0$. It follows that

$$\log(S_T) = \log(F_0) + \frac{1}{F_0}(S_T - F_0) - \int_0^{S_T} \frac{(K - F_0)^+}{K^2} dK - \int_{S_T}^{\infty} \frac{(F_0 - K)^+}{K^2} dK$$

- Accordingly,

$$\log\left(\frac{S_T}{F_0}\right) = \frac{1}{F_0}(S_T - F_0) - \int_0^{S_T} \frac{(K - F_0)^+}{K^2} dK - \int_{S_T}^{\infty} \frac{(F_0 - K)^+}{K^2} dK. \quad (7)$$

Risk Neutral Pricing

→ The pricing formulas for options according to the first principle of $\mathbb{Q}$F.

$$c_0(K) = e^{-r_0 T}\mathbb{E}_0^{\mathbb{Q}}\left[\left(S_T - K\right)^+\right]; \tag{8}$$

$$p_0(K) = e^{-r_0 T}\mathbb{E}_0^{\mathbb{Q}}\left[\left(K - S_T\right)^+\right]. \tag{9}$$

Risk Neutral Expectation

- Under the risk neutral measure,

$$\mathbb{E}_0^{\mathbb{Q}}(S_T) = F_0$$

- Consequently,

$$\mathbb{E}_0^{\mathbb{Q}}\left(\frac{1}{F_0}(S_T - F_0)\right) = \frac{1}{F_0}\left[\mathbb{E}_0^{\mathbb{Q}}(S_T) - F_0\right] = 0.$$

- Therefore, under the risk-neutral measre $\mathbb{Q}$, (7) becomes

$$\mathbb{E}_0^{\mathbb{Q}}\left[\log\left(\frac{S_T}{F_0}\right)\right] = -e^{r_0 T}\int_{F_0}^{\infty}\frac{c_0(K)}{K^2}\,dK - e^{r_0 T}\int_0^{F_0}\frac{p_0(K)}{K^2}\,dK. \quad (10)$$

Variance as a Difference of Two Returns

- Pre-U math suggests that

$$\log(1 + r_t) = r_t - \frac{1}{2} r_t^2 + O(r_t^3).$$

- In other words, the following approximation is a good one because at daily frequency or higher, the asset return r_t is generally very small.

$$r_t^2 \approx 2\big[r_t - \log(1 + r_t)\big].$$

- Since the mean $\mathbb{E}(r_t) \approx 0$, $\mathbb{E}(r_t^2) \approx \mathbb{V}(r_t)$, i.e. the variance.
- Insight: Twice the difference between the simple return r_t and the log return $\log(1 + r_t)$ is the variance.
- Is it guaranteed that $r_t - \log(1 + r_t) \geq 0$?

Integrated Variance

- Integrate with respect to time from $t = 0$ to $t = T$.

$$\int_0^T \sigma_t^2 \, dt = 2 \int_0^T r_t \, dt - 2 \int_0^T \log(1 + r_t) dt.$$

- Under the risk neutral measure, according to the first principle of $\mathbb{Q}$F,

$$\mathbb{E}_0^{\mathbb{Q}} \left(\int_0^T r_t \, dt \right) = \int_0^T \mathbb{E}_0^{\mathbb{Q}} (r_t) dt = \int_0^T r_0 \, dt = r_0 T,$$

where r_0 is the risk-free rate associated to the risk-neutral measure.

Integrated Variance as Model-Free Variance

- On the other hand, because $\log\left(\frac{S_1}{S_0}\right) + \log\left(\frac{S_2}{S_1}\right) + \cdots$ $\cdots + \log\left(\frac{S_{T-1}}{S_{T-2}}\right) + \log\left(\frac{S_T}{S_{T-1}}\right) = \log\left(\frac{S_T}{S_0}\right)$, we obtain

$$\mathbb{E}_0^{\mathbb{Q}}\left(\int_0^T \log(1+r_t)\,dt\right) = \mathbb{E}_0^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_0}\right)\right].$$

- Putting all terms together, we have

$$\sigma^2_{\text{MF}}T := \mathbb{E}_0^{\mathbb{Q}}\left(\int_0^T \sigma_t^2\,dt\right) = 2r_0T - 2\mathbb{E}_0^{\mathbb{Q}}\left[\log\left(\frac{S_T}{S_0}\right)\right]. \quad (11)$$

Formula of Model-Free Variance

- Finally, we write

$$\log \frac{S_T}{S_0} = \log \frac{S_T}{F_0} + \log \frac{F_0}{S_0}.$$

- Substituting this result and (10) into (11), we obtain

$$\sigma^2_{\mathsf{MF}} T = 2r_0 T + 2e^{r_0 T}\left(\int_{F_0}^{\infty} \frac{c_0}{K^2}\, dK + \int_0^{F_0} \frac{p_0}{K^2}\, dK\right) - 2\log \frac{F_0}{S_0}\,.$$

- Since $F_0 = e^{r_0 T} S_0$, the first and last terms cancel out. Accordingly,

$$\boxed{\sigma^2_{\mathsf{MF}} T = 2e^{r_0 T}\left(\int_{F_0}^{\infty} \frac{c_0}{K^2}\, dK + \int_0^{F_0} \frac{p_0}{K^2}\, dK\right).} \qquad (12)$$

Features of Model-Free Approach

- No requirement for an option pricing model
  $\Longrightarrow$ No model risk!
- No worry about parameters
  - The only exogenous inputs are risk-free interest rate and dividend yields
- No bias
  - σ_{MF} reflects volatility across all out-of-the-money strike prices and thus reflects the option skew
- Uses both put and call options
  $\Longrightarrow$ σ_{MF} is less sensitive to individual option prices.
- The formula is beautiful!

Advantages and Limitation

- The model-free approach incorporates information from out-of-the-money puts and calls (with respect to the forward price F_0) to produce a *single* implied volatility σ_{MF} for a given maturity.
- Given the weight $\dfrac{1}{K^2}$, out-of-the-money puts contribute more to σ_{MF}, hence "fear" gauge.
- The model-free approach to implied volatility is applicable only for European options.
- Equity index options are typically European but stock options are American.

Issues in Implementation

- Strike price is not continuous but discrete.
- Strike prices in the option chain are not from 0 to ∞.
- Most options are illiquid and most have only ask prices but not bid prices.

Market Reality

Calls			Puts	
Bid	Ask	Strike	Bid	Ask
28	32.8	40	0	4.8
23	27.8	45	0	4.8
20.5	25.3	47.5	0	4.8
18	22.8	50	0	4.8
15.5	20.3	52.5	0	4.8
13.1	17.9	55	0	4.8
10.6	15.4	57.5	0	4.8
8.1	12.9	60	0	4.8
5.6	10.4	62.5	0	4.8
3.2	8	65	0	4.8
1.15	5.8	67.5	0	4.8
0.05	4.8	70	0	4.8
0	4.8	72.5	0.25	4.8
0	4.8	75	2.15	6.9
0	4.8	77.5	4.6	9.4
0	4.8	80	7.1	11.9
0	4.8	82.5	9.6	14.4
0	4.8	85	12.1	16.9
0	4.8	90	17.1	21.9
0	5	95	21.5	26.5
0	5	100	26.5	31.5

- Option chain of BKX, KBW Nasdaq Bank Index (@ 70.61)
- Expiration: Sep 15, 2016
- Source: Optionetics, as of Aug 22, 2016

- Discrete strike price
- Limited strike range
  - lowest strike ℓ=$40
  - highest strike h=$100
- Not liquid
- But quotes are firm, ready for trades

Takeaways

- Implied volatility used to be model-dependent.
  - Black-Scholes option pricing formula
  - Binomial tree

- Model risk
  - All models are wrong....

    — George Box
  - A smile surface that extends well into the wings, which are suspect of model risk

- Model-free approach to implied volatility
  - VIX — discrete computations
  - Academics — "smooth" computations

Week 7 Assignment

Suppose the option price curve can be modeled over any small sub-interval $(K_i, K_{i+1}]$ a cubic spline:

$$o^i = s_1^i K^3 + s_2^i K^2 + s_3^i K + s_4^i,$$

where s_1^k to s_4^k are the spline coefficients. Applying your pre-U math, show that integration over each sub-interval $(K_i, K_{i+1}]$ admits a closed form expression:

$$\int_{K_i}^{K_{i+1}} \frac{o^i}{K^2}\,dK = s_1^i \frac{K_{i+1}^2 - K_i^2}{2} + s_2^i \big(K_{i+1} - K_i\big) + s_3^i \ln\left(\frac{K_{i+1}}{K_i}\right) - s_4^i \left(\frac{1}{K_{i+1}} - \frac{1}{K_i}\right).$$

Week 7 Additional Exercises

The asset price S_T at maturity T, which is unknown at time $t = 0$, can potentially attain a low value denoted by L, or appreciate substantially to a high value H. Show that

$$\int_{F_0}^{S_T} \frac{S_T - X}{X^2}\, dX = \int_{F_0}^{H} \frac{(S_T - X)^+}{X^2}\, dX + \int_{L}^{F_0} \frac{(X - S_T)^+}{X^2}\, dX\,.$$